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Flatness, LCM-Stability, and Related Module-Theoretic Properties

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The relationship between flatness and LCM-stability is clarified by the following two results. A finitely generated ideal I of an integral domain is flat if and only if I is n-flat for some integer $n \geqslant 2$. There exists an integral domain with a nonflat ideal $J = (a, b, c)$ such that $Jab \cap Jac \cap Jbc = J(ab, ac, bc)$. Next, related module-theoretic properties, (**) and (*), respectively weaker than projectivity and flatness, are introduced. Under appropriate finiteness conditions, these properties are preserved by certain inverse limits. Their study leads to new characterizations of quasi-complete local rings and coherent integral domains.

1. INTRODUCTION

This article is a sequel to [2, 6]. All rings are assumed commutative and all modules unital.

Sections 2 and 3 investigate further the relationship between flatness and LCM-stability. Recall that a torsion-free module E over an integral domain R is said to be LCM-stable over R in case $E(Ra \cap Rb) = Ea \cap Eb$ for all $a, b \in R$. (This property was first studied in [7, 12] in case E is an extension domain of R.) Although flatness implies LCM-stability, Uda [12] has shown that the converse is false. However, [6, Corollary 2.3] established that, for 2-generated ideals I of an integral domain R, a certain weak version of LCM-stability is actually equivalent to flatness (and hence, if $I \neq 0$, to invertibility). Our first result, Example 2.1, shows that there is no

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analogous characterization of flatness for 3-generated ideals. On a positive note, Theorem 3.5 establishes that, for finitely generated ideals of an integral domain, LCM-stability *is* equivalent to flatness.

Indeed, more is shown. First, recall that a module E over an integral domain R is called n-flat (over R) in case each linear relation of n elements $r_1, ..., r_n \in R$ with coefficients in E is a linear consequence of linear relations of the r_i's with coefficients in R. It is known that flatness is equivalent to "n-flat for each positive integer $n \geqslant 2$" and that 2-flatness is equivalent to LCM-stability [6, Theorem 3.3(b)]. Now, we can state a sharper form of Theorem 3.5: a finitely generated ideal I of an integral domain R is R-flat if (and only if) I is n-flat over R for some positive integer $n \geqslant 2$. Also of some independent interest in Section 3 is Proposition 3.1, which connects n-flatness and tensor product.

Sections 4 and 5 take their inspiration from the characterizations in [2] of invertible and flat ideals. This work had been the principal motivation for the study of LCM-stability in [6]. We abstract these characterizing properties to the module-theoretic setting, obtaining properties called (**) and (*). (See Section 4 for their definitions.) It turns out that (**) (resp., (*)) is implied by, but is not equivalent to, projectivity (resp., flatness). The main result of Section 4, Theorem 4.5, states that if R is a local ring, then its completion $\hat{R}$ satisfies (**) as an R-module if and only if R is quasi-complete (in the sense of [8, 1]). The main result of Section 5 is that, under appropriate finiteness conditions but not in general, (**) and (*) are each preserved by arbitrary products. (See Theorem 5.3 and Proposition 5.1(c).) Section 5 also includes a new (*)-theoretic characterization of coherent integral domains (in Corollary 5.6) and new proofs of some known properties of such integral domains.

2. A Noncriterion for Flatness

It was shown in [6, Corollary 2.3] that a 2-generated ideal $I = Ra + Rb$ of an integral domain R is R-flat if (and only if) $Ia \cap Ib = I(Ra \cap Rb)$. We show next that there is no analogous criterion for flatness of 3-generated ideals.

Example 2.1. Let X, Y, Z be commuting, algebraically independent indeterminates over a field k. Set $R = k[X, Y, Z]$ and $I = RXY + RXZ + RYZ$. Then

$$IXY \cap IXZ \cap IYZ = I(RXY \cap RXZ \cap RYZ)$$

although I is not R-flat.

Proof. We claim that $I^{-1} = R$. Indeed, if $\delta \in I^{-1}$, write $\delta = fg^{-1}$ for relatively prime polynomials f, g in the UFD R. As $fI \subset gR$, it follows that

g is a factor of each of XY, XZ, and YZ, whence g is a constant. Thus $\delta \in R$, as claimed. Hence $II^{-1} = I \neq R$, so that I is not invertible. Alternatively, notice that I, though finitely generated, is not principal. As R is factorial, it follows that I is not invertible. Being finitely generated and nonzero, I is therefore not flat.

Put $J = IXY \cap IXZ \cap IYZ$ and $K = I(RXY \cap RXZ \cap RYZ)$. Evidently $K \subset J$. We shall show $J \subset K$. View J as

$$J = (X^2Y^2, X^2YZ, XY^2Z) \cap (X^2YZ, X^2Z^2, XYZ^2) \cap (XY^2Z, XYZ^2, Y^2Z^2).$$

As $K = IXYZ$, it is possible, via a long but straightforward argument exploiting the factoriality of R, to show $J \subset K$. We leave the details for the reader. For a more elegant argument, one may use the fact that X, Y, Z is an R-sequence to conclude that J is generated by monomials of the form $X^\alpha Y^\beta Z^\gamma$, where α, β, $\gamma \geqslant 1$ and at most one of α, β, γ is 1. As X^2Y^2Z, X^2YZ^2, $XY^2Z^2 \in J$ it follows that each such $X^\alpha Y^\beta Z^\gamma$ is in K, whence $J \subset K$. ∎

3. Finitely Generated LCM-Stable Ideals Are Flat

The following useful result connects n-flatness and tensor products.

Proposition 3.1. *Let M be an n-flat module over an integral domain R, for some positive integer n. If I is an n-generated ideal of R, then the canonical homomorphism $g: I \otimes_R M \to M$ is a monomorphism.*

Proof. We shall show that each $e \in \ker(g)$ is 0. To this end, write $I = Ri_1 + \cdots + Ri_n$, and note that $e = \sum i_j \otimes m_j$, for suitable $m_1, ..., m_n \in M$ such that $\sum i_j m_j = g(e) = 0$. As M is n-flat over R, there exist elements $r_{kj} \in R$ and $m_j^* \in M$ such that

$$m_k = \sum_j r_{kj} m_j^* \qquad \text{for} \quad k = 1, ..., n$$

and

$$\sum_{k=1}^{n} i_k r_{kj} = 0 \qquad \text{for each } j.$$

Then

$$e = \sum_k i_k \otimes \Big(\sum_j r_{kj} m_j^*\Big) = \sum_j \Big(\sum_k i_k r_{kj} \otimes m_j^*\Big)$$
$$= \sum 0 \otimes m_j^* = \sum 0 = 0. \quad ∎$$

Remark 3.2. Calculations similar to the one above are given, for $n = 2$, in the proofs of [6, Corollary 3.4 and Theorem 3.8]. Moreover, with the aid of a standard criterion for flatness [3, Proposition 1, p. 37], one sees easily that Proposition 3.1 leads to a new proof of [6, Corollary 3.4].

We next infer a companion for [6, Theorem 2.2].

COROLLARY 3.3. *Let I be an n-generated ideal of an integral domain R, for some positive integer n. Then I is R-flat if (and only if) I is n-flat over R.*

Proof. As I is n-flat, Proposition 3.1 assures that the multiplication map $I \otimes_R I \to I$ is a monomorphism. Then, by [5, Proposition 1], I is R-flat. ∎

Remark 3.4. The preceding result is best-possible. Indeed, a 1-flat 2-generated ideal I of an integral domain R need not be R-flat. To see this, observe that if an integral domain R is not a Prüfer domain, then some (necessarily 1-flat) 2-generated ideal of R is not R-flat.

The main result of this section is

THEOREM 3.5. *Let I be a finitely generated ideal of an integral domain R. Then the following conditions are equivalent:*

(1) *I is LCM-stable over R;*

(2) *I is n-flat over R for some positive integer $n \geqslant 2$;*

(3) *the canonical homomorphism $I \otimes_R J \to J$ is a monomorphism for each 2-generated ideal J of R such that $J \subset I$;*

(4) *I is R-flat.*

Proof. (4) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): It is straightforward to verify that if $m < n$ are positive integers and a module E is n-flat over R, then E is m-flat over R. As LCM-stability is equivalent to 2-flatness, the assertion follows.

(1) $\Rightarrow$ (3): By (1), I is 2-flat over R and so, by Proposition 3.1, $J \otimes_R I \to JI$ is monomorphism. As $J \otimes_R I \cong I \otimes_R J$, the assertion follows.

(3) $\Rightarrow$ (4): Without loss of generality, R is quasi-local. It is enough to show that I is a principal ideal of R. Deny. Consider a minimal generating set $S = \{a, b, ...\}$ for I as an ideal of R, with $a \neq b$. Set $J = Ra + Rb$. By minimality, $\nu(J) = 2$. (As usual, $\nu(E)$ denotes the minimum cardinality of a generating set for an R-module E.) Moreover, by (3), $I \otimes_R J \cong IJ$, and so $\nu(I \otimes_R J) = \nu(IJ)$. As $\nu(I \otimes_R J) = \nu(I)\, \nu(J)$, it follows that

$$\begin{aligned} 2\nu(I) = \nu(Ia + Ib) &= |\{xy \colon x \in S,\ y \in \{a, b\}| \\ &\leqslant 2\,|S| - 1 \qquad (\text{since } ab = ba) \\ &= 2\nu(I) - 1, \end{aligned}$$

the desired contradiction. ∎

Remark 3.6. (a) For future applications, it may be helpful to make the following observation, in the spirit of [5, Proposition 2]. Formally as before, one can define what is meant by n-flatness of a module over an arbitrary (commutative) ring. Then, by the above proof that $(3) \Rightarrow (4)$, we have the following result. If I is a finitely generated 2-flat ideal of a (commutative) ring R, then I is locally principal over R.

(b) The implication $(3) \Rightarrow (4)$ in Theorem 3.5 establishes the converse of Proposition 3.1 in case M is a finitely generated ideal of R and $n = 2$. It would be interesting to have additional instances of the converse of Proposition 3.1.

4. On Intersections and Quasi-completeness

This section treats the module-theoretic properties, dubbed $(*)$ and $(**)$ in the following, which underlay the ideal-theoretic studies in [2]. Let R be a ring and let E be an R-module. We shall say that E *satisfies* $(*)$ (over R) if $(I \cap J)E = IE \cap JE$ for all ideals I, J of R, and that E *satisfies* $(**)$ (over R) if $(\bigcap I_\alpha)E = \bigcap (I_\alpha E)$ for all families $\{I_\alpha\}$ of ideals of R. By adapting the proofs of [2, Theorems 2 and 1], we easily have

PROPOSITION 4.1. *Let R be an integral domain and E a torsion-free R-module. Then*:

(a) *E satisfies $(*)$ if and only if E is R-flat.*

(b) *Suppose, in addition, that E is a fractional ideal of R. Then E satisfies $(**)$ if and only if E is R-projective.*

Remark 4.2. The above properties are related in general as follows. If R is a ring and E is an R-module, then

$$\begin{array}{ccc} E \text{ is } R\text{-projective} & \Rightarrow & E \text{ satisfies } (**) \text{ over } R \\ \Downarrow & & \Downarrow \\ E \text{ is } R\text{-flat} & \Rightarrow & E \text{ satisfies } (*) \text{ over } R. \end{array}$$

Proof. The "vertical" implications are clear. Moreover, as in the proof of [2, Theorem 2, $(3) \Rightarrow (1)$], it follows via [3, Proposition 6, p. 17] that flatness implies $(*)$. Finally, to show that each projective R-module E satisfies $(**)$, choose P so that $E \oplus P$ is a free R-module F. Then

$$\bigcap (I_\alpha E) \oplus \bigcap (I_\alpha P) = \bigcap (I_\alpha E \oplus I_\alpha P) = \bigcap (I_\alpha F).$$

Expressing F in terms of an R-basis shows that this intersection is just $(\bigcap I_\alpha)F$, and hence is just $(\bigcap I_\alpha)E \oplus (\bigcap I_\alpha)P$. Equating first summands as submodules of E gives $\bigcap (I_\alpha E) = (\bigcap I_\alpha)E$, completing the proof. ∎

The above result has natural applications. For instance, if a commutative R-algebra S is free as an R-module, then S satisfies $(**)$ over R. In particular, for each positive integer n, the polynomial ring $R[X_1, ..., X_n]$ satisfies $(**)$ over R. The same holds for the formal power series ring $R[[X]]$ if R is Noetherian: this will follow from Theorem 5.3(b) below.

By Proposition 4.1(a), an overring S of an integral domain R satisfies $(*)$ over R if and only if S is R-flat. We show next that the situation for $(**)$ is very different.

THEOREM 4.3. *If an overring S of an integral domain R satisfies $(**)$ over R, then $S = R$.*

Proof. Assume that S satisfies $(**)$. Then, by Remark 4.2 and Proposition 4.1(a), S is R-flat. Thus, by [11, Lemma 1 and Theorem 1], $(b:a)S = S$ for all nonzero $a, b \in R$ such that $ab^{-1} \in S$. (As usual, $(b:a)$ denotes $\{r \in R: ra \in Rb\}$.) Let $I = \bigcap (b:a)$, the intersection being indexed by the elements $ab^{-1} \in S$. By hypothesis, $IS = \bigcap ((b:a)S)$, which is just $\bigcap S = S$. In particular, $I \neq 0$; pick a nonzero element c of I. Now, for each $ab^{-1} \in S$, $c \in (b:a)$, whence $ab^{-1} \in Rc^{-1}$; thus $S \subset Rc^{-1}$. As S is then a fractional ideal of R satisfying $(**)$, Proposition 4.1(b) yields that S is invertible. Hence S is finitely generated as an R-module. It follows that S is integral over R. However, there are no flat integral proper overrings [11, Proposition 2], so $S = R$, completing the proof. ∎

Despite the preceding result, some important algebras (which need not be free as modules) *do* satisfy $(**)$. We shall show this in Examples 4.4 and Theorem 4.5. To motivate that work, consider the above concepts for a special R. It is easy to see that if R is a valuation domain (more generally, a ring whose ideals are linearly ordered by inclusion), then *each* R-module E satisfies $(*)$; and that if R is a DVR with local uniformizing parameter π, then an R-module E satisfies $(**)$ if and only if $\bigcap \pi^n E = 0$. These considerations lead naturally to the DVR $\mathbb{Z}_{p\mathbb{Z}}$, which appears next in Example 4.4(a).

The following examples give extra motivation for Theorems 4.5 and 5.3(b).

EXAMPLES 4.4. (a) Despite Proposition 4.1(b) and Remark 4.2, $(**)$ does not imply projectivity, even for a torsion-free module over a DVR. For instance, let p be a prime positive integer, let $R = \mathbb{Z}_{p\mathbb{Z}}$, and let $E = \hat{R}$. View E as the integral domain $\mathbb{Z}_p$ of p-adic integers. It can be shown that E satisfies $(**)$ over R. (To see this, one may apply the above comment to the DVR $\widehat{\mathbb{Z}_{p\mathbb{Z}}}$, since $\bigcap p^n \widehat{\mathbb{Z}_{p\mathbb{Z}}} = 0$; or one may appeal to Theorem 4.5.) As R may be viewed as a subdomain of E, E is a torsion-free R-module; however, E is not free as an R-module (cf. [9, Theorem 18]).

(b) The following additional example that "(**) $\not\Rightarrow$ projectivity" will be generalized in Section 5. Let $R = \mathbb{Z}$ and let $E = \prod \mathbb{Z}$, the product of denumerably many copies of $\mathbb{Z}$. It can be shown that E satisfies (**) over R (via a direct calculation using the definition of (**) and standard facts about $\mathbb{Z}$, or via Theorem 5.3(b)). Of course, E is torsion-free over R; however, by a celebrated result of Baer (cf. [9]), E is not a free R-module. ∎

To set the stage for a generalization of Example 4.4(a), consider (R, M), a local (Noetherian) ring R with maximal ideal M. Let $(\hat{R}, \hat{M})$ denote the M-adic completion of R. It is well known that $\hat{R}$ is R-flat; hence, by Remark 4.2, $\hat{R}$ satisfies (*) over R. Before determining when $\hat{R}$ satisfies (**) over R, it is convenient to recall the following material (cf. [8, 1]). We say that (R, M) is *quasi-complete* if, for each descending sequence $B_1 \supset B_2 \supset \cdots$ of ideals of R and each integer k, there exists an integer $m(k)$ such that $B_{m(k)} \subset (\bigcap B_n) + M^k$. Equivalently, R is quasi-complete if the assignment $I \mapsto I\hat{R}$ gives a lattice-isomorphism $L(R) \to L(\hat{R})$ from the lattice $L(R)$ of ideals of R to the lattice $L(\hat{R})$ of ideals of $\hat{R}$. Completeness of R (that is, the condition $R \overset{\cong}{\to} \hat{R}$) implies quasi-completeness, but the converse is false. Indeed, if R is a one-dimensional (still local) integral domain, then R is quasi-complete if and only if $\hat{R}$ is an integral domain. Thus, for instance, $\mathbb{Z}_{p\mathbb{Z}}$ and $\mathbb{Q}[X]_{(X)}$ are each quasi-complete but not complete.

The main result of this section is

THEOREM 4.5. *Let (R, M) be a local ring. Then $\hat{R}$ satisfies (**) over R if and only if R is quasi-complete.*

Proof. The "if" part follows since the (complete) lattice-isomorphism $L(R) \to L(\hat{R})$ preserves arbitrary intersections. Conversely, let $\hat{R}$ satisfy (**). Consider a descending sequence $\{B_n\}$ of ideals of R and let k be a positive integer. Since $\hat{R}$ is (quasi-) complete, there exists a positive integer $m(k)$ such that

$$B_{m(k)}\hat{R} \subset \bigcap (B_n\hat{R}) + \hat{M}^k.$$

However, $\bigcap (B_n\hat{R}) = (\bigcap B_n)\hat{R}$ since $\hat{R}$ satisfies (**). Using standard facts about completions (cf. [10, Proposition 6, p. 95]), we find

$$\begin{aligned} B_{m(k)} = B_{m(k)}\hat{R} \cap R &\subset \left(\left(\bigcap B_n\right)\hat{R} + \hat{M}^k\right) \cap R \\ &= \left(\left(\bigcap B_n\right)\hat{R} + M^k\hat{R}\right) \cap R = \left(\left(\left(\bigcap B_n\right) + M^k\right)\hat{R}\right) \cap R \\ &= \left(\bigcap B_n\right) + M^k. \end{aligned}$$

Hence R is quasi-complete, as asserted. ∎

5. On Direct Products and Coherence

Motivated in part by Example 4.4(b), we devote this section to studying when a product of modules satisfies (**). By also considering the analogous question for (*), we shall find that applications to coherent rings arise naturally.

It is convenient next to recall the following material from [4, Theorems 2.1 and 2.2]. A ring R is said to be *coherent* in case each finitely generated ideal of R is finitely presented. (The most familiar examples of coherent rings are arbitrary Noetherian rings and arbitrary Prüfer domains.) Equivalently, R is coherent in case each of the following three equivalent conditions holds: if $\{M_\beta\}$ is a set of flat R-modules, then $\prod M_\beta$ is R-flat; each product of (arbitrarily many) copies of R is R-flat; $I \cap J$ is finitely generated for each pair of finitely generated ideals I and J of R, and the annihilator $(0:a) = \{r \in R: ra = 0\}$ is finitely generated for each $a \in R$. This section's techniques will lead to new proofs of some of these facts in case R is an integral domain, as well as giving a new characterization of coherent integral domains (see Corollary 5.6).

The next result collects some basic information.

Proposition 5.1. *Let R be a ring and $\{E_\beta\}$ a set of R-modules. Set $S = \bigoplus E_\beta$ and $P = \prod E_\beta$. Then:*

(a) *S satisfies (*) (resp., (**)) if and only if E_β satisfies (*) (resp., (**)) for each β.*

(b) *If P satisfies (*) (resp., (**)), then E_β satisfies (*) (resp., (**)) for each β.*

(c) *If E_β satisfies (**) (or (*)) for each β, then P need not satisfy (*) (or (**)).*

Proof. (a) For each ideal I of R, $IS = I \sum E_\beta = \sum IE_\beta = \bigoplus IE_\beta$. Thus, for ideals I_α of R, one has

$$\left(\bigcap I_\alpha\right) S = \bigoplus_\beta \left(\left(\bigcap_\alpha I_\alpha\right) E_\beta\right)$$

and

$$\bigcap (I_\alpha S) = \bigcap \left(\bigoplus I_\alpha E_\beta\right) = \bigoplus_\beta \left(\bigcap_\alpha (I_\alpha E_\beta)\right).$$

Thus, $(\bigcap I_\alpha) S = \bigcap (I_\alpha S)$ if and only if $(\bigcap I_\alpha) E_\beta = \bigcap (I_\alpha E_\beta)$ for each β. The assertions follow easily.

(b) Apply (a), noting that $P = E_\beta \oplus (\prod \{E_\gamma : \gamma \neq \beta\})$.

(c) Consider any integral domain R which is not coherent. By the above-cited work of Chase, some product $P = \prod R$ of copies of R is not R-flat. As P is torsion-free over R, Proposition 4.1(a) and Remark 4.2 yield that P does not satisfy $(*)$ and, hence, that P does not satisfy $(**)$. However (each copy of) R is R-projective, and hence satisfies both $(**)$ and $(*)$. The proof is complete. ▮

We shall also require the following result, which is rather well known.

LEMMA 5.2. *Let R be a ring and I an ideal of R such that $\nu(I) = \gamma$. Then the following conditions are equivalent*:

(1) *I is finitely generated.*

(2) *For each collection $\{E_\beta\}$ of R-modules, the canonical inclusion map $I \prod E_\beta \to \prod IE_\beta$ (of submodules of $\prod E_\beta$) is the identity.*

(3) *There exists an indexed family $\{E_\beta\}$ of copies of R (that is, $E_\beta = R$ for each β), with index set having cardinality at least γ, such that $I \prod E_\beta = \prod (IE_\beta)$* (that is, such that $I \prod R = \prod I$).

Theorem 5.3 is the main result in this section. It is important for at least three reasons. First, it reduces the assertion (in Example 4.4(b)) that $\mathbb{Z} \times \mathbb{Z} \times \cdots$ satisfies $(**)$ over $\mathbb{Z}$ to the obvious assertion that $\mathbb{Z}$ (is Noetherian and) satisfies $(**)$ over itself. Second, Theorem 5.3(b) implies, as promised prior to Theorem 4.3, that if R is a Noetherian ring, then $R[[X]]$ satisfies $(**)$ over R: one need only note that $R[[X]] \cong R \times R \times \cdots$ as R-modules and that R satisfies $(**)$ over itself. Third, the result is sharp, for both its parts would fail without their "coherent" and "Noetherian" hypotheses: see the example in Proposition 5.1(c).

THEOREM 5.3. *Let R be a ring, $\{E_\beta\}$ a collection of R-modules, and $P = \prod E_\beta$. Then*:

(a) *If R is coherent and if E_β satisfies $(*)$ over R for each β, then P satisfies $(*)$ over R.*

(b) *If R is Noetherian and if E_β satisfies $(**)$ over R for each β, then P satisfies $(**)$ over R.*

Proof. We begin with a useful observation: in the definition of "satisfies $(*)$" (resp., $(**)$), the test ideals I, J (resp., I_α) may be assumed finitely generated. For such data, $I \cap J$ is finitely generated in part (a) by coherence, and $\bigcap I_\alpha$ is finitely generated in part (b) since R is Noetherian. A common proof for (a) and (b) can now be given, in terms of an indexed family $\{I_\gamma\}$ of finitely generated ideals, viewed as $\{I, J\}$ in part (a) and as $\{I_\alpha\}$ in part (b). By combining the above remarks about finite generation, condition (2) in Lemma 5.2, the hypotheses on E_β, and commutativity of $\bigcap$ and $\prod$, we have

$$\left(\bigcap I_\gamma\right)P=\prod_\beta\left(\left(\bigcap_\gamma I_\gamma\right)E_\beta\right)=\prod_\beta\left(\bigcap_\gamma (I_\gamma E_\beta)\right)=\bigcap_\gamma\left(\prod_\beta (I_\gamma E_\beta)\right).$$

Again invoking condition (2) in Lemma 5.2, we may rewrite this intersection as $\bigcap (I_\gamma P)$, completing the proof. ∎

Remark 5.4. Theorem 5.3 (and its proof) may be generalized as follows. Let R be a ring and let $\{E_\beta\}$ be an inverse system of R-modules such that $I \varprojlim E_\beta \xrightarrow{\cong} \varprojlim IE_\beta$ for each finitely generated ideal I of R. Set $E=\varprojlim E_\beta$. Then if R is coherent (resp., Noetherian) and E_β satisfies $(*)$ (resp., $(**)$) for each β, then E satisfies $(*)$ (resp., $(**)$).

This offers an alternate proof of the "if" part of Theorem 4.5. For let (R, M) be quasi-complete. Then R/M^k satisfies $(**)$ over R for each positive integer k. Moreover, for each finitely generated ideal I of R, one has

$$\begin{aligned} I \varprojlim R/M^k &= I\hat{R} \cong I \otimes_R \hat{R} \cong \hat{I} = \varprojlim I/IM^k \\ &\cong \varprojlim I/M^k \cap I \cong \varprojlim (I+M^k)/M^k = \varprojlim I(R/M^k). \end{aligned}$$

Thus $\hat{R}=\varprojlim R/M^k$ satisfies $(**)$, as asserted. ∎

We next begin the promised reworking of parts of [4].

PROPOSITION 5.5. *Let R be a ring and γ a cardinal number such that each ideal of R has a generating set with cardinality at most γ. Suppose that there exists an R-flat product $P=\prod_\beta R$ of copies of R, with index set having cardinality $\beta \geqslant \gamma$. Then*:

(a) *$I \cap J$ is finitely generated for each pair I, J of finitely generated ideals of R.*

(b) *$(0:a)$ is finitely generated for each $a \in R$.*

Proof. (a) Since P is R-flat, it satisfies $(*)$ by Remark 4.2, whence $(I \cap J)P = IP \cap JP$. Since I and J are each finitely generated, condition (2) of Lemma 5.2 gives $IP=\prod I$ and $JP=\prod J$, all products being indexed by β elements. It follows that $IP \cap JP=\prod (I \cap J)$, whence $(I \cap J)\prod R = \prod (I \cap J)$. Since $\beta \geqslant \gamma$, the implication $(3) \Rightarrow (1)$ in Lemma 5.2 yields the assertion.

(b) $K=(0:a)=\ker(m_a)$, where $m_a\colon R \to R$ is multiplication by a. By flatness of P, $K \otimes_R P$ is identified with $\ker(m_a \otimes \mathrm{id}_P)$; that is, after applying the isomorphism $R \otimes_R P \xrightarrow{\cong} P$, with the kernel of multiplication by a on P.

The upshot is that $K \prod R = \prod K$, each product being indexed by β elements. Since $\beta \geqslant \gamma$, the implication $(3) \Rightarrow (1)$ in Lemma 5.2 may again be invoked, showing that K is finitely generated, thus completing the proof. ∎

COROLLARY 5.6. *An integral domain R is coherent if (and only if) there exist cardinal numbers $\gamma \leqslant \beta$ such that each ideal of R has a generating set with cardinality at most γ and the product $P = \prod_\beta R$ of β copies of R satisfies $(*)$ over R.*

Proof. The "only if" part follows from the criterion in [4, Theorem 2.1(a)] since flatness implies $(*)$. For the "if" part, one need only reread the proof of Proposition 5.5(a) and appeal to the coherence criterion that the intersection of any two finitely generated ideals of R be finitely generated. ∎

Our final result gives an alternate proof that products preserve flatness over a coherent integral domain.

PROPOSITION 5.7. *Let R be a (coherent) integral domain such that the intersection of any pair of finitely generated ideals of R is finitely generated. Let $\{E_\beta\}$ be a collection of R-flat modules E_β. Then $P = \prod E_\beta$ is R-flat.*

Proof. Being flat, each E_β is torsion-free over R, and hence so is P. Thus, by Proposition 4.1(a), it suffices to prove that P satisfies $(*)$; that is, by the first observation in the proof of Theorem 5.3, that $(I \cap J)P = IP \cap JP$ for all finitely generated ideals I, J of R. Since $I \cap J$ is finitely generated by hypothesis and since each E_β (being flat) satisfies $(*)$, we may repeat the calculations displayed in the proof of Theorem 5.3, yielding $(I \cap J)P = (\prod IE_\beta) \cap (\prod JE_\beta)$. Finally, since I and J are finitely generated, Lemma 5.2 reduces this intersection to $IP \cap JP$, completing the proof. ∎

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